

SI 2014 Q1

(i) B1 for $3 = 2^2 - 1^2$
 $5 = 3^2 - 2^2$
 $8 = 3^2 - 1^2$
 $12 = 4^2 - 2^2$
 $16 = 5^2 - 3^2$

(ii) B1 for $2n+1 \equiv (n+1)^2 - n^2$ or $2n-1 \equiv n^2 - (n-1)^2$

(iii) B1 for $(k+1)^2 - (k-1)^2 \equiv 4k$

(iv) APPROACH 1 - parity argument (4 cases to consider)

odd² - even² and even² - odd² = odd B1 B1
ALT. $a^2 - b^2$ is even iff a, b both odd or both even B2

odd² - odd² = $(2a+1)^2 - (2b+1)^2$ M1 attempted
= $4a^2 + 4a - 4b^2 - 4b$ A1
or $4(a^2 + a - b^2 - b)$

even² - even² = $(2a)^2 - (2b)^2 = 4a^2 - 4b^2$ A1
M1 or $4(a^2 - b^2)$

∴ Every even difference of 2 squares is of the form $4k$
and not $4k+2$ E1 MUST BE STATED; NOT TO BE
GIVEN UNLESS THEY HAVE FULLY SHOWN IT SO.

APPROACH 2 - modular arithmetic

$x^2 \pmod{4} \equiv 0, 1$ M2 A1
⇒ $a^2 - b^2 \pmod{4} \equiv 1, 0, -1 (\equiv 3)$ A1 A1 A1
So $a^2 - b^2 \not\equiv 2 \pmod{4}$ E1 MUST BE STATED etc.

$$(v) \quad m^2 - n^2 = (m+n)(m-n) = pq$$

[M1] Diff. of 2 sqs. factsn.
($p > q$ here)

$$\Rightarrow p = m+n, \quad q = m-n \quad [A1]$$

OR

$$pq = m+n, \quad 1 = m-n \quad [A1]$$

Since p, q (odd) primes, these are the only two factorisations of pq
 $\Rightarrow$ exactly two ways [E1]

If $q = 2$ we have $2p = m^2 - n^2$ where p is odd

so this is of the form $4h+2$ [B1]

and (iv) $\Rightarrow$ this is impossible [E1]

6

$$(vi) \quad 675 = 3^3 \cdot 5^2 \quad [M1] \text{ Prime factorisation attempt } [A1] \text{ correct}$$

$$\Rightarrow (3+1)(2+1) = 12 \text{ factors } [M1]$$

$$\Rightarrow 6 \text{ factor-pairs } [A1]$$

ALT for 2nd M1 A1	Listing factors	675	225	135	75	45	27
		1	3	5	9	15	25

PLEASE BE VERY STRICT WITH THESE LAST 4 MARKS.

FOR FINAL A1 THEY MUST INDICATE CLEARLY

THAT THEY KNOW THE REQUIRED ANSWER IS 6

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NB $675 = (m+n)(m-n) = ab \Rightarrow m+n = a, m-n = b$
 $\Rightarrow m = \frac{1}{2}(a+b), n = \frac{1}{2}(a-b)$

and so the six possibilities are

$$675 = 338^2 - 337^2 = 114^2 - 111^2 = 70^2 - 65^2$$

$$= 42^2 - 33^2 = 30^2 - 15^2 = 26^2 - 1^2$$

NOT ASKED-FOR

MO MO for "search" methods, even if all 6 found, without proper justification

SI 2014 Q2

(i) Method 1 - differentiation

$$y = -(2-x) \cdot \ln(2-x) + (2-x) + c \quad (x < 2)$$

$$\Rightarrow \frac{dy}{dx} = -(2-x) \cdot \frac{1}{2-x} \cdot -1 + \ln(2-x) \cdot 1 - 1$$

[M1] Good use of the PRODUCT RULE

[M1] Good use of the CHAIN RULE

[A1] unsimplified

$$= 1 + \ln(2-x) - 1$$

$$= \ln(2-x) \quad [A1] \text{ all correct and simplified}$$

Method 2 - integration

$$\int \ln(2-x) \cdot 1 \, dx \quad [M1] \text{ Use of parts} \quad (x < 2)$$

$$= x \cdot \ln(2-x) - \int \frac{-1}{2-x} \cdot x \, dx \quad [A1]$$

$$= x \ln(2-x) - \int \frac{2-x-2}{2-x} \, dx$$

$$= x \ln(2-x) + \int \left(-1 + \frac{2}{2-x}\right) \, dx \quad [M1]$$

$$= x \ln(2-x) - x - 2 \ln(2-x) + C_1$$

$$= -(2-x) \ln(2-x) + (2-x) + C \quad (C = C_1 - 2)$$

[A1] Correctly found GIVEN ANSWER

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$$(iii) \text{ Area} = \int_0^{\sqrt{3}} \ln(2-x) \, dx + \int_0^{\sqrt{3}} \ln(2+x) \, dx \quad [M1] \text{ Splitting up log. term}$$

$$= \left[-(2-x) \ln(2-x) - x \right]_0^{\sqrt{3}} + \left[(2+x) \ln(2+x) - x \right]_0^{\sqrt{3}}$$

[A1] with/without +2 [A1] ditto the 2

$$= \left(-(2-\sqrt{3}) \ln(2-\sqrt{3}) - \sqrt{3} + 2 \ln 2 \right) + \left((2+\sqrt{3}) \ln(2+\sqrt{3}) - \sqrt{3} - 2 \ln 2 \right)$$

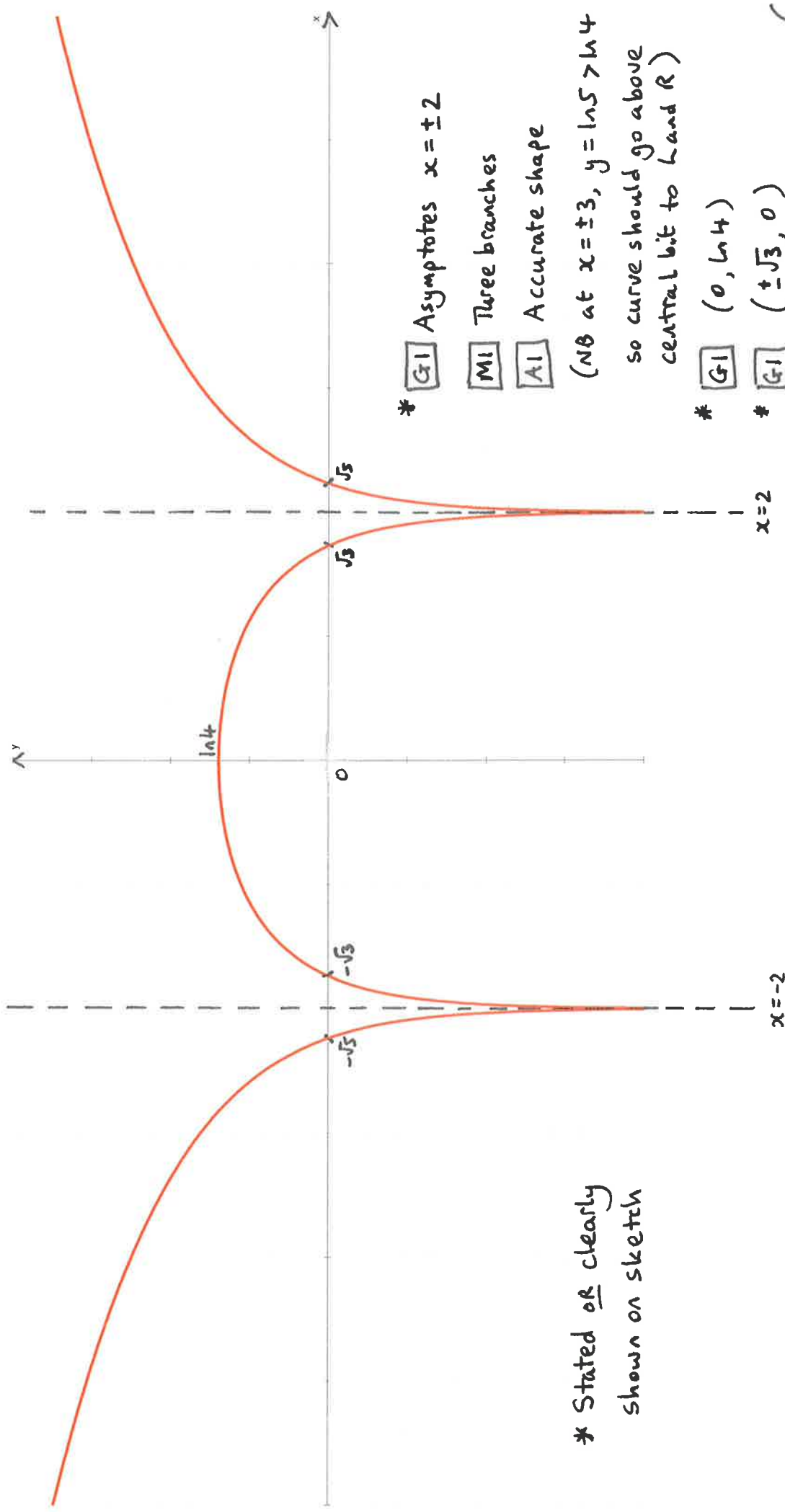
$$= (2-\sqrt{3}) \ln(2+\sqrt{3}) - \sqrt{3} + (2+\sqrt{3}) \ln(2+\sqrt{3}) - \sqrt{3}$$

$$= 4 \ln(2+\sqrt{3}) - 2\sqrt{3} \quad [A1] \text{ GIVEN ANSWER correctly obtained}$$

* [M1] Use of $2-\sqrt{3} = (2+\sqrt{3})^{-1}$

MUST APPEAR EXPLICITLY

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* Stated or clearly shown on sketch

* G1 Asymptotes $x = \pm 2$

M1 Three branches

A1 Accurate shape

(NB at $x = \pm 3$, $y = \ln 5 > \ln 4$
so curve should go above central bit to L and R)

* G1 $(0, \ln 4)$

* G1 $(\pm\sqrt{3}, 0)$

* G1 $(\pm\sqrt{5}, 0)$

$$\begin{aligned}
 \text{(iv) 2nd area} &= - \int_{\sqrt{3}}^2 \left\{ \ln(2-x) + \ln(2+x) \right\} \\
 &= \left[(2-x) \ln(2-x) + x \right]_{\sqrt{3}}^2 - \left[(2+x) \ln(2+x) - x \right]_{\sqrt{3}}^2 \quad \boxed{M1} \pm \text{here} \\
 &= \left(0^* + 2 - (2-\sqrt{3}) \ln(2-\sqrt{3}) - \sqrt{3} \right) \\
 &\quad * \text{ using } t \ln t \rightarrow 0 \text{ as } t \rightarrow 0 \quad \boxed{A1} \\
 &\quad - \left(4 \ln 4 - 2 - (2+\sqrt{3}) \ln(2+\sqrt{3}) + \sqrt{3} \right) \quad \boxed{A1} \\
 &= 4 \ln(2+\sqrt{3}) + 4 - 2\sqrt{3} - 4 \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \text{Total area is } & 2 \left\{ \left(4 \ln(2+\sqrt{3}) - 2\sqrt{3} \right) + \left(4 \ln(2+\sqrt{3}) + 4 - 2\sqrt{3} - 4 \ln 4 \right) \right\} \\
 & \quad \boxed{M1} \quad \text{(iii)'s GIVEN ANSWER} \quad \text{ft above provided signs now correct and} \\
 & \quad \ln(2-\sqrt{3}) \rightarrow -\ln(2+\sqrt{3}) \quad \text{sorted}
 \end{aligned}$$

$$\underline{\underline{\text{Area} = 8 \left(2 \ln(2+\sqrt{3}) + 1 - \sqrt{3} - \ln 4 \right)}} \quad \boxed{A1}$$

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SI 2014 Q3

$$(i) LHS = \left[\frac{x^3}{3} \right]_0^b = \frac{1}{3} b^3 \quad [B1] \quad \text{and} \quad RHS = \left(\left[\frac{x^2}{2} \right]_0^b \right)^2 = \frac{1}{4} b^4 \quad [B1]$$

$$b \neq 0 \Rightarrow \underline{b = 4/3} \quad [A1] \quad \text{CORRECT ANSWER ONLY}$$

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$$(ii) \quad \frac{b^3-1}{3} = \left(\frac{b^2-1}{2} \right)^2 \quad (M1)$$

$$\Rightarrow 4(b-1)(b^2+b+1) = 3(b-1)^2(b+1)^2 \quad [M1] \quad \text{creating a quadratic (or cubic)}$$

$$\Rightarrow 3b^4 - 4b^3 - 6b^2 + 7 = 0$$

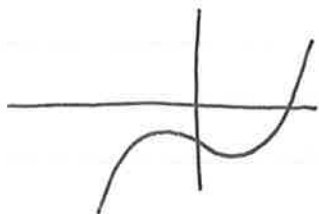
$$\Rightarrow (b-1)(3b^3 - b^2 - 7b - 7) = 0$$

$$b > a \text{ so } b \neq 1 \text{ and } \underline{3b^3 - b^2 - 7b - 7 = 0} \quad [A1] \quad \text{GIVEN ANSWER properly justified}$$

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$$\text{For } y = 3x^3 - x^2 - 7x - 7, \quad \frac{dy}{dx} = 9x^2 - 2x - 7 = (9x+7)(x-1) \quad [M1]$$

$$\text{giving TPs at } \left(-\frac{7}{9}, -\frac{864}{243}\right) \text{ and } (1, -12) \quad [A1] \quad \text{y-coords not required provided they can be shown to be -ve}$$



Both TPs below x-axis

$\Rightarrow$ exactly one real (positive) root

[E1] Explanation

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ALT: $\frac{dy}{dx} = \left(3x - \frac{1}{3}\right)^2 - \frac{64}{9} > 0$ for $x > 1$ so $\exists$ no TPs to right of $x=1$

$$y(2) = -1 \text{ and } y(3) = 44 \quad (B1)$$

"Change-of-Sign" Rule (for a continuous fn.) $\Rightarrow 2 < \text{root} < 3$ [E1]

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(iii) In general, we have $\frac{b^3-a^3}{3} = \left(\frac{b^2-a^2}{2}\right)^2$

$$\Rightarrow 4(b-a)(b^2+ab+a^2) = 3(b-a)^2(b+a)^2 \quad \boxed{M1} \text{ MUST factorise}$$

$$b > a \Rightarrow q \neq 0 (=b-a) \Rightarrow 4(p^2-ab) = 3qp^2 \quad \boxed{M1} \text{ } p, q \text{ substd.} \\ (p=a+b)$$

$$\text{NB } p^2 - q^2 = 4ab \quad \Rightarrow \quad 4p^2 - (p^2 - q^2) = 3qp^2 \quad \boxed{A1}$$

$$\Rightarrow \quad \underline{\underline{3p^2 + q^2 = 3p^2q}}$$

GIVEN ANSWER
legitimately
obtained

$$\Rightarrow \quad \underline{\underline{p^2 = \frac{q^2}{3(q-1)}}} \quad \boxed{B1} \text{ CAO}$$

Since $p^2 > 0$, $q > 1$ from this $\boxed{E1}$

$$\text{Also, } p^2 - q^2 \geq 0 \Rightarrow \frac{q^2}{3(q-1)} - \frac{q^2 \cdot 3(q-1)}{3(q-1)} = \frac{q^2(4-3q)}{3(q-1)} \geq 0 \quad \boxed{E1} \quad \boxed{M1}$$

$$\Rightarrow q = b-a \leq \frac{4}{3} \quad \boxed{A1}$$

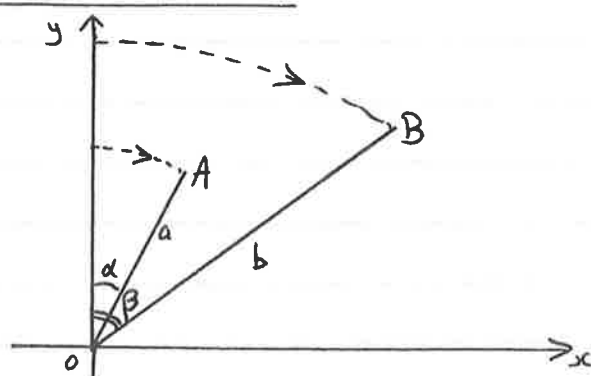
$$\text{ALT } p \geq q \text{ (and both +ve)} \Rightarrow \frac{q^2}{3(q-1)} \geq q^2 \Rightarrow \frac{1}{3} \geq q-1 \Rightarrow \frac{4}{3} \geq q$$

(Hence $1 < b-a \leq \frac{4}{3}$, as required)

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SI 2014 Q4



Hour-hand A moves at $\frac{2\pi}{12}$ rad/hr

[M1] Set-up

Minute-hand B 2π

so at time t hours after noon (w.l.o.g. start with both hands together at noon)

$$A = (a \sin \alpha, a \cos \alpha) \quad B = (b \sin \beta, b \cos \beta)$$

Where $\alpha = kt$

and

$$\beta = 12kt$$

[A1]

$$k = \frac{\pi}{6}$$

$$\begin{aligned} \text{Then } AB^2 &= (b \sin 12kt - a \sin kt)^2 + (b \cos 12kt - a \cos kt)^2 \quad [M1] \quad [A1] \\ &= b^2 + a^2 - 2ab \left\{ \sin 12kt \sin kt + \cos 12kt \cos kt \right\} \quad \text{correct unsimplified} \\ &\quad \text{use } \cos(P-Q) \quad [M1] \end{aligned}$$

$$\Rightarrow \underline{\underline{D = (b^2 + a^2 - 2ab \cos 11kt)^{1/2}}}$$

[A1]

ALT. By the Cosine Rule 6

[M1] Diff^g D

[M1] with good effort at use of CHAIN RULE

$$\text{Rate of change is } \frac{dD}{dt} = \frac{1}{2} (b^2 + a^2 - 2ab \cos 11kt)^{-1/2} \cdot 2ab \sin 11kt \cdot 11k$$

$$= 11kab \left\{ \frac{\sin 11kt}{(b^2 + a^2 - 2ab \cos 11kt)^{1/2}} \right\}$$

[A1] Okay unsimplified

(Allow FT if only constant factors are incorrect)

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This is a maximum when $\frac{d^2 D}{dt^2} = 0$ [M1]

NB Just did this already

i.e. (ignoring denominator)

$$(b^2 + a^2 - 2ab \cos 11kt)^{1/2} \cdot 11k \cos 11kt = \sin 11kt \cdot \frac{11kab \sin 11kt}{(b^2 + a^2 - 2ab \cos 11kt)^{1/2}}$$

[M1] QUOTIENT AND [A1] CHAIN RULES APPLIED

i.e. When $(b^2 + a^2 - 2ab \cos 11kt) \cos 11kt = ab \sin^2 11kt$

$$\Leftrightarrow (b^2 + a^2)C = ab(2C^2 + S^2) = ab(1 + C^2)$$

$$\Leftrightarrow abC^2 - (b^2 + a^2)C + ab = 0$$
 [M1]

$$\Leftrightarrow (aC - b)(bC - a) = 0$$
 [M1]

Since $b > a$, $\cos 11kt \neq \frac{b}{a}$ so $\cos 11kt = \frac{a}{b}$ [E1] Explanation of choice for C

* and $D = (b^2 + a^2 - 2ab \cdot \frac{a}{b})^{1/2} = (b^2 - a^2)^{1/2}$ [A1] Allow even if EO above

GIVEN ANSWER

When $b = 2a$, $\cos \frac{11\pi t}{6} = \frac{1}{2}$ [M1] [A1]

$$\Rightarrow \frac{11\pi t}{6} = \frac{\pi}{3} \quad (1^{st} \text{ occurrence})$$
 [M1]

$$\Rightarrow t = \frac{2}{11} \text{ hours} = \frac{120}{11} = 10\frac{10}{11} \approx 11 \text{ mins}$$
 [A1]

GIVEN ANSWER

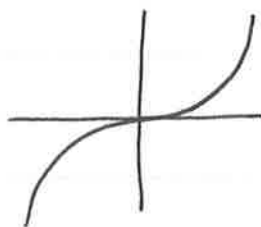
* Note I am not expecting candidates to show it is a MAX. point rather than a MIN.

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SI 2014 Q5

(i) If $a=0$, $f(x) = x^3$

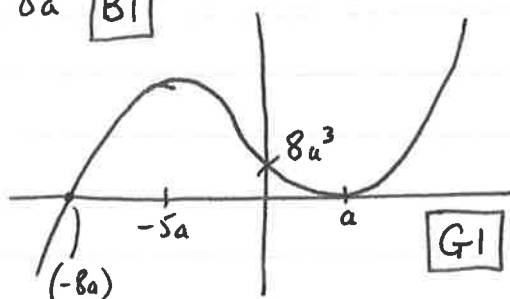


and $f \geq 0 \forall x \geq 0$

[B1]

If $a > 0$, $f'(x) = 3(x+2a)^2 - 27a^2 = 3\{(x+2a-3a)(x+2a+3a)\}$
= 0 at $(a, 0)$ and $(-5a, 108a^3)$ [M1] [A1]

Also, $f(0) = 8a^3$ [B1]



giving $f(x) \geq 0 \forall x \geq 0$

[A1]

Conclusion
MUST be
noted

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ALT. $f(x) = (x-a)^2(x+8a)$ gives zeroes at $x = -8a$, $x = a$ (twice) and positive cubic shape $\Rightarrow$ etc. QED. [M1] [A1] [G1]

(ii) With $a=y$, (i) gives $27xy^2 \leq (x+2y)^3$
[M1] [A1]

Using $x+2y \leq 3$ [M1] $\Rightarrow \underline{xy^2 \leq 1}$ [A1]

Equality from (i) occurs when $x = a (=y)$ [M1]
and $x+2y = 3$ gives equality iff $\underline{x=y=1}$ [A1]

Give B1 for correct result from missing/inconclusive/fudged working

(iii) From (i), $(x+2a)^3 - 27a^2x \geq 0 \quad \forall x \geq 0$

Set $x=p$ and $2a = q+r$ [M1]

$$\Rightarrow (p+q+r)^3 - 27\left(\frac{q+r}{2}\right)^2 p \geq 0 \quad [A1]$$

* Now $\left(\frac{q+r}{2}\right)^2 \geq qr \Leftrightarrow (q-r)^2 \geq 0$ which is clearly true
[M1] Attempt to prove this [A1] Validly done

giving $(p+q+r)^3 \geq 27pqr$ [A1]

NB Some may simply cite $AM \geq GM$ to justify this.

Anyone who ignores request to "use part (i)....." and uses the 3-variable AM-GM: $\frac{p+q+r}{3} \geq \sqrt[3]{pqr}$

$$\Rightarrow (p+q+r)^3 \geq 27pqr$$

gives M1 A1 only

From (i), the equality case arises when
 $x = a$ i.e. $p = \frac{q+r}{2}$

[E1] Both bits must be supplied

and $\left(\frac{q+r}{2}\right)^2 \geq qr$ equality iff $q=r$

giving $p=q=r$ [B1]

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SI 2014 Q6

(i) $u_{n+1} = 4u_n(1-u_n)$ $u_0 = u$
 $u_0 = \sin^2 \theta \Rightarrow u_1 = 4 \sin^2 \theta (1 - \sin^2 \theta)$ [M1]
 $= 4 \sin^2 \theta \cos^2 \theta = \sin^2 2\theta$ [A1]
 $\Rightarrow u_2 = \sin^2 4\theta$ similarly [A1]

Conjecture: $u_n = \sin^2(2^n \theta)$ [B1]
 $\Rightarrow u_{n+1} = \sin^2(2 \cdot 2^n \theta)$ from above [M1]
 $= \sin^2(2^{n+1} \theta)$ [A1] and the result follows by induction

NB No need to provide baseline case or explain the inductive method.

(ii) $v_{n+1} = -p v_n^2 + q v_n + r$ * (p, q, r constants; $p \neq 0$)
 and $v_0 = v$

[M1] Substn. of $v_n = \alpha u_n + \beta$ into *
 $v_{n+1} = \alpha u_{n+1} + \beta = -p(\alpha^2 u_n^2 + 2\alpha\beta u_n + \beta^2) + q(\alpha u_n + \beta) + r$
 $= -p\alpha^2 u_n^2 + (q\alpha - 2p\alpha\beta)u_n + (q\beta + r - p\beta^2)$ [A1]

$\Rightarrow u_{n+1} = -p\alpha u_n^2 + (q - 2p\beta)u_n + \frac{(q-1)\beta + r - p\beta^2}{\alpha}$
 [M1]

c.f. $-4u_n^2 + 4u_n$ [M1]

$\Rightarrow \underline{p\alpha = 4}$ $\underline{q - 2p\beta = 4}$ [A1] and $\underline{(q-1)\beta + r = p\beta^2}$ [A1]

Substg. $\alpha = \frac{4}{p}$ and $\beta = \frac{q-4}{2p}$ into [M1]

$\Rightarrow \frac{(q-1)(q-4)}{2p} + r = \frac{p(q-4)^2}{4p^2}$

$\Rightarrow 2(q-1)(q-4) + 4pr = (q-4)^2$

$\Rightarrow 4pr = (q-4) \cdot -(q-2)$

i.e. $\underline{4pr = 8 + 2q - q^2}$ [A1]

GIVEN ANSWER
legitimately
obtained

$$v_0 = v_1 = 1 \text{ and } v_{n+1} = -v_n^2 + 2v_n + 2 \quad (p=1, q=2, r=2)$$

$$\text{Check: LHS} = 4pr = 8 \text{ and RHS} = 8 + 2 \cdot 2 - 2^2 = 8 \text{ also } \boxed{B1}$$

$$\text{Use substn. } v_n = 4u_n - 1 \quad \boxed{M1} \text{ with } u_0 = \frac{1}{2}$$

$$\Rightarrow u_0 = \sin^2 \frac{\pi}{4} \quad \boxed{M1} \text{ giving soln. } u_n = \sin^2 \left(2^n \frac{\pi}{4} \right) \quad \boxed{M1} \text{ Use of (i)'s result}$$

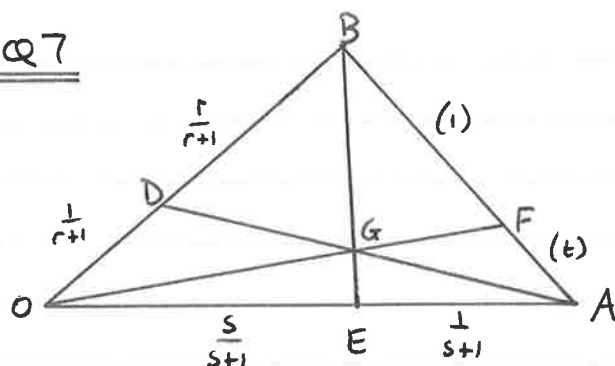
$$\Rightarrow \underline{\underline{v_n = 4 \sin^2 \left(2^n \frac{\pi}{4} \right) - 1}} \quad \boxed{A1}$$

$$\text{Sequence is } \{1, 3, -1, -1, -1, \dots\} \quad \boxed{B1}$$

$$\text{since } v_n = -1 \quad \forall n \geq 2$$

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SI 2014 Q7



$$\begin{aligned}\vec{OA} &= \underline{a} \\ \vec{OB} &= \underline{b} \\ \vec{OG} &= \underline{g}\end{aligned}$$

(i) $\underline{d} = \left(\frac{1}{r+1}\right) \underline{b}$ [B1] and $\underline{e} = \left(\frac{s}{s+1}\right) \underline{a}$ [B1]

Eqn. AD is $\underline{r} = \underline{a} + \alpha(\underline{d} - \underline{a})$ or $(1-\alpha)\underline{a} + \alpha\underline{d}$ [M1]
i.e. $\underline{r} = (1-\alpha)\underline{a} + \left(\frac{\alpha}{r+1}\right)\underline{d}$ [A1]

Eqn. BE is $\underline{r} = \underline{b} + \beta(\underline{e} - \underline{b})$ or $(1-\beta)\underline{b} + \beta\underline{e}$ [M1]
i.e. $\underline{r} = (1-\beta)\underline{b} + \left(\frac{\beta s}{s+1}\right)\underline{a}$ [A1]

AD $\cap$ BE when $1-\alpha = \frac{\beta s}{s+1}$ and $1-\beta = \frac{\alpha}{r+1}$ [M1] equating coeffs

[M1] Solving simultaneously for α, β in terms of r, s

* See below

$\Rightarrow \alpha = \frac{1+r}{1+r+rs}$ and $\beta = \frac{r(1+s)}{1+r+rs}$ [A1] [A1]

(OR $1-\alpha = \frac{rs}{1+r+rs}$ AND/OR $1-\beta = \frac{1}{1+r+rs}$ depending on their choices in the line eqns.)

giving $\underline{g} = \left(\frac{rs}{1+r+rs}\right)\underline{a} + \left(\frac{1}{1+r+rs}\right)\underline{b}$ [A1] GIVEN ANSWER legitimately obtained

* [M1] Sensible method e.g. $1-\alpha + \alpha = \frac{\beta s}{s+1} + (1-\beta)(r+1)$ for eliminating one of α, β

$\Rightarrow s+1 = \beta s + (rs+r+s+1) - \beta(rs+r+s+1)$

$\Rightarrow \beta = \dots$

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(ii) F on $OG \Rightarrow \underline{f} = \underline{\lambda g}$ [B1]

F divides AB in ratio $t:1 \Rightarrow \underline{f} = \left(\frac{1}{t+1}\right)\underline{a} + \left(\frac{t}{t+1}\right)\underline{b}$ [M1] [A1]

[M1] Equating terms

$$\frac{\lambda rs}{1+r+rs} = \frac{1}{t+1} \quad \text{and} \quad \frac{\lambda}{1+r+rs} = \frac{t}{t+1} \quad [A1] [A1]$$

Dividing (e.g.) $\Rightarrow \underline{\underline{t = \frac{1}{rs}}}$ [M1] [A1]

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SI 2014 Q8

(i) L_a is $y - 0 = \left(\frac{1-a}{-a}\right)(x-a)$ [M1] [A1]

and L_b is $y = \left(\frac{1-b}{-b}\right)(x-b)$ similarly [B1] ft

Lines meet when $\left(1-\frac{1}{a}\right)x + 1-a = \left(1-\frac{1}{b}\right)x + 1-b$ [M1]

since $b \neq a$,

$$b-a = \left(\frac{1}{a} - \frac{1}{b}\right)x = \left(\frac{b-a}{ab}\right)x$$

$$\Rightarrow \underline{x = ab}$$
 [A1]

$$\underline{y = (1-a)(1-b)}$$
 [A1]

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(ii) As $b \rightarrow a$, $P \rightarrow (a^2, (1-a)^2)$ [B1]

i.e. $x = a^2$, $y = (1-a)^2$

Since $0 < a < 1$, $a = \sqrt{x}$ [M1] and $\underline{y = (1-\sqrt{x})^2}$ [A1]

and $0 < \sqrt{x} < 1 \Rightarrow 0 < x < 1$ [E1]

GIVEN ANSWER

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(iii) $\frac{dy}{dx} = 2(1-\sqrt{x}) \cdot -\frac{1}{2}x^{-1/2} = \frac{(1-\sqrt{x})}{-\sqrt{x}}$

[M1] Diffg.

[M1] Good use
of CHAIN RULE

[A1] Correct unsimplified

3

Eqn. of tgt. to C is $y - \underline{(1-\sqrt{c})^2} = \left(\frac{1-\sqrt{c}}{-\sqrt{c}}\right)(x - \underline{c})$
at $(c, (1-\sqrt{c})^2)$

[M1] Attempt at tgt. provided all in terms
do NOT involve x

[A1] correct unsimplified

$$y - 1 + 2\sqrt{c} - c = \left(\frac{1 - \sqrt{c}}{-\sqrt{c}} \right) x + \sqrt{c} - c$$

$$y = \left(\frac{1 - \sqrt{c}}{-\sqrt{c}} \right) x + (1 - \sqrt{c}) \quad \boxed{A1} \text{ Suitably simplified form}$$

$$\text{c.f. } L_a \text{ (e.g.) } y = \left(\frac{1 - a}{-a} \right) x + (1 - a) \quad \boxed{M1}$$

$$\text{so tgt. to } C \text{ is } L_{\sqrt{c}} \quad \boxed{A1} \quad \text{where } 0 < \sqrt{c} < 1 \quad \boxed{EI} \quad 6$$

SI 2014 Q9

$$\left(\begin{array}{l} \ddot{x} = -kg \\ \ddot{y} = -g \end{array} \quad \begin{array}{l} \dot{x} = u \cos \theta - kgt \\ \dot{y} = u \sin \theta - gt \end{array} \quad \begin{array}{l} x = ut \cos \theta - \frac{1}{2} kgt^2 \\ y = ut \sin \theta - \frac{1}{2} gt^2 \end{array} \right)$$

At max. ht. $\dot{y} = 0 \Rightarrow T_H = \frac{u \sin \theta}{g}$ [M1] [A1]

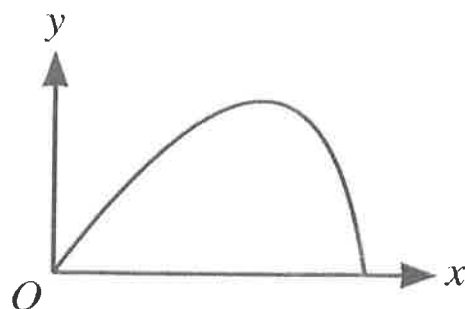
At landing, $y = 0$ ($t \neq 0$) $\Rightarrow T_L = \frac{2u \sin \theta}{g}$ [M1] [A1]

$T = \frac{u \cos \theta}{kg}$ is the time when $\dot{x} = 0$ [B1]

$= \frac{u \sin \theta}{g} \cdot \frac{1}{k \tan \theta}$ [B1]

Case 1 $k \tan \theta < \frac{1}{2} \Rightarrow T_H < T_L < T$ [B1]

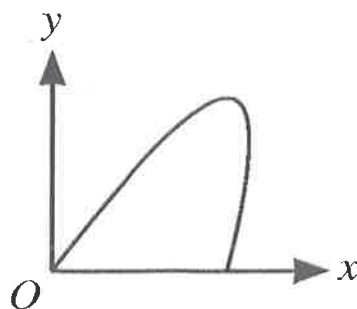
[G1] Parabola-like shape



[G1] "Shortened" parabola on RHS

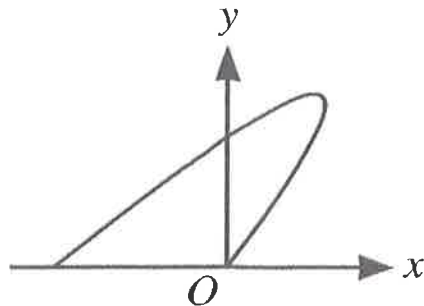
[G1] Clearly never vertical

Case 2 $\frac{1}{2} < k \tan \theta < 1 \Rightarrow T_H < T < T_L$ [B1]



[G2]

Case 3 $k \tan \theta > 1 \Rightarrow T < T_H < T_L$ B1



G2

3

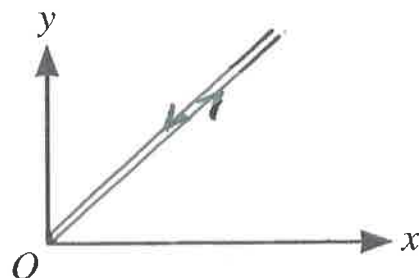
When $k \tan \theta = 1$, $\ddot{x} = \ddot{y} = 0$ together
particle lands at O

$\therefore$ moves in a straight line

since resultant accl. is $\parallel$ direction of projection E1

B1
B1
B1 }

*



* or G3

4

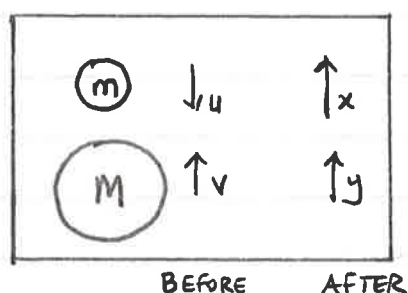
SI 2014 Q10

- (i) Ball's centre of mass falls H , so M1
By energy (e.g.) $u^2 = 2gH$ at ground A1
NEL $v = eu$ B1

By energy (e.g.) $v^2 = 2g(H_1 - R)$ B1

Then $2g(H_1 - R) = e^2 \cdot 2gH \Rightarrow H_1 = R + e^2 H$ A1 M1 6

(ii)



NB $u = \sqrt{2gH}$ and $v = eu = e\sqrt{2gH}$

B1 This needs to be used at some stage.

CLM $\uparrow$ $mx + My = Meu - mu$ M1 A1

NEL $x - y = e(u + eu)$ M1 A1

Solving (for x ; y not needed) M1
$$\begin{aligned} mx + My &= Meu - mu \\ Mx - My &= Meu + Me^2u \end{aligned}$$

Adding: $(m+M)x = u(Me^2 + 2Me - m)$

$$x = \frac{(Me^2 + 2Me - m)u}{M+m}$$
 A1

7

By energy (e.g.) smaller ball's centre of mass rises a distance d ,
given by $x^2 = 2gd$ [M1]

$$\Rightarrow d = \frac{1}{2g} \left(\frac{Me^2 + 2Me - m}{M+m} \right)^2 \cdot 2gH \quad [A1]$$

$$\left(= H \left(\frac{M(e+1)^2}{M+m} - 1 \right)^2 \right)$$

and $h = 2R + r + d$ [M1]

Subst^g $R = 0.2$, $r = 0.05$, $H = 1.8$, $h = 4.5$ and $e = \frac{2}{3}$ [M1]

$$d = 1.8 \left(\frac{\frac{4}{9}M + \frac{4}{3}M - m}{M+m} \right)^2 = 1.8 \left(\frac{\frac{16}{9}\lambda - 1}{\lambda + 1} \right)^2 \quad \lambda = \frac{M}{m} \quad [M1]$$

and $4.5 = 0.45 + 1.8 \left(\frac{\frac{16}{9}\lambda - 1}{\lambda + 1} \right)^2$

or $10 = 1 + 4 \left(\frac{\frac{16}{9}\lambda - 1}{\lambda + 1} \right)^2$

$$\Rightarrow \frac{\frac{16}{9}\lambda - 1}{\lambda + 1} = \frac{3}{2} \quad \left[\text{condone failure to consider } -\frac{3}{2} \right]$$

(it gives $\lambda < 0$)

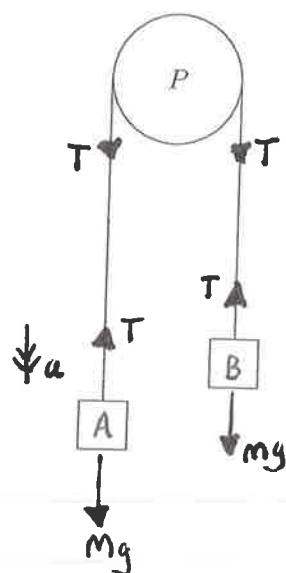
[M1] Solving a suitable linear eqn for λ

$$\Rightarrow \underline{\underline{\lambda = \frac{M}{m} = 9}} \quad [A1] \text{ CAO}$$

7

SI 2014 Q11

(i)



N2L ↓ A $Mg - T = Ma$ [B1]

N2L ↑ B $T - mg = ma$ [B1]

Adding: $(M-m)g = (M+m)a$ [M1]

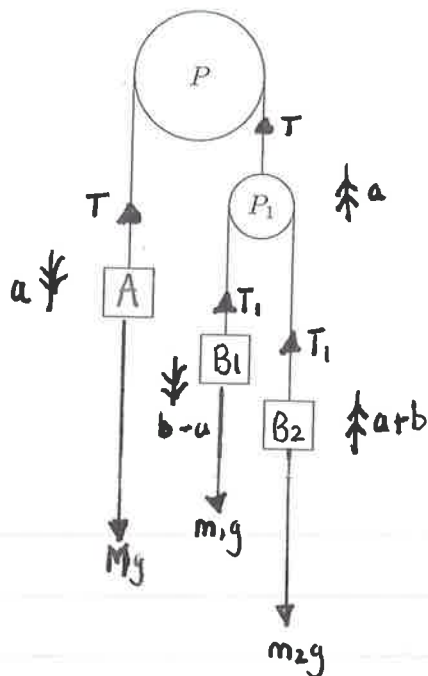
$\Rightarrow a_1 = \frac{(M-m)g}{M+m}$ [A1] GIVEN ANSWER

Substⁿ for T and force on pulley ↓ is 2T [M1]

$\Rightarrow \frac{4Mmg}{M+m}$ [A1]

6

(ii)



Let b be accln. of B₁, B₂ relative to P₁

N2L ↓ A $Mg - T = Ma$ [B1] (1)

N2L ↓ B₁ $m_1g - T_1 = m_1(b-a)$ [B1] (2)

N2L ↑ B₂ $T_1 - m_2g = m_2(a+b)$ [B1] (3)

For P₁ $T = 2T_1$ [B1] (4)

+ [B1] at some stage for sorting out "global" acclns. of B₁, B₂

5

e.g.

$$m_2.(2): m_1 m_2 g - m_2 T_1 = m_1 m_2 b - m_1 m_2 a$$

[M1] Suitable elimination (for a) process used

$$m_1.(3): m_1 T_1 - m_1 m_2 g = m_1 m_2 a + m_1 m_2 b$$

$$\text{Subtracting } \uparrow: (m_1 + m_2) T_1 - 2m_1 m_2 g = 2m_1 m_2 a$$

$$T = M(g - a) \Rightarrow T_1 = \frac{1}{2} M(g - a)$$

[M1] T, T_1 eliminated

$$\Rightarrow \frac{1}{2} M(g - a)(m_1 + m_2) - 2m_1 m_2 g = 2m_1 m_2 a$$

$$\Rightarrow M(m_1 + m_2)g - 4m_1 m_2 g = M(m_1 + m_2)a + 4m_1 m_2 a$$

$$\Rightarrow a = \left(\frac{M(m_1 + m_2) - 4m_1 m_2}{M(m_1 + m_2) + 4m_1 m_2} \right) g$$

[M1] "a" found in this form

$$a_2 = \left(\frac{M - 4\mu}{M + 4\mu} \right) g$$

$$\text{Where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

[A1]

GIVEN ANSWER

4

$$\frac{a_1}{g} = \frac{M - m}{M + m}$$

$$\text{and } \frac{a_2}{g} = \frac{Mm - 4m_1 m_2}{Mm + 4m_1 m_2}$$

[M1]

$$\text{so } a_1 = a_2 \Leftrightarrow (M - m)(Mm + 4m_1 m_2) = (M + m)(Mm - 4m_1 m_2)$$

[M1]

$$\Leftrightarrow \underline{M^2 m + 4Mm_1 m_2 - Mm^2} - 4m_1 m_2 = \underline{M^2 m - 4Mm_1 m_2 + Mm^2} - 4m_1 m_2$$

$$\Leftrightarrow 8Mm_1 m_2 = 2Mm^2$$

$$\Leftrightarrow 4m_1 m_2 = (m_1 + m_2)^2$$

$$\Leftrightarrow 0 = (m_1 - m_2)^2$$

[M1] OR AM-GM

$$\Leftrightarrow m_1 = m_2$$

[A1]

[E1] If they have made both directions of proof clear (clearly stated)

5

SI 2014 Q12

		Coin	
		H	T
Die	1	$ k-1 \frac{3}{12}$	$ k-1 \frac{3}{12}$
	2	$ 2k-1 \frac{2}{12}$	$ k-2 \frac{2}{12}$
	3	$ 3k-1 \frac{1}{12}$	$ k-3 \frac{1}{12}$

[B1] All 6 outcomes

[B1] correct accompanying probabilities

2

$$E(X^2) = \frac{6}{12}(k-1)^2 + \frac{2}{12}(2k-1)^2 + \frac{2}{12}(k-2)^2 + \frac{1}{12}(3k-1)^2 + \frac{1}{12}(k-3)^2$$

[M1] for $\sum x^2 p(x)$

[B1] using $| \dots |^2 = (\dots)^2$ throughout

$$= \frac{1}{12} \left\{ 6(k^2 - 2k + 1) + 2(4k^2 - 4k + 1) + 2(k^2 - 4k + 4) + (9k^2 - 6k + 1) + (k^2 - 6k + 9) \right\}$$

[A1] correct to here

$$= \frac{1}{6} (13k^2 - 20k + 13)$$

$$= k + \frac{13}{6}(k-1)^2$$

[A1] GIVEN ANSWER

legitimately obtained

[M1] Method for determining when this is an integer

$$\text{N.B. } E(X^2) = k + 2(k-1)^2 + \frac{1}{6}(k-1)^2$$

[A1] so $k = 1, 7, (13, \dots)$

6

$$E(X) = \frac{6}{12}|k-1| + \frac{2}{12}|2k-1| + \frac{2}{12}|k-2| + \frac{1}{12}|3k-1| + \frac{1}{12}|k-3|$$

[B1]

If $k=1$, $E(X) = \frac{2}{3} \notin \mathbb{N}$

[B1] shown

If $k=7$, $E(X) = 8$

[B1]

so $k=7$

3

Prob. Distribn. of X
(for $k=7$)

x	4	5	6	13	20
$P(X=x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{6}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

[B1] x 's

[B1] p 's

2

$$\begin{aligned}
 P(X > 25) &= P(20+20, 20+13, 13+20, 20+6, 6+20, 13+13) \\
 &= \frac{1}{144} + 2 \cdot \frac{2}{144} + 2 \cdot \frac{6}{144} + \frac{4}{144} \\
 &= \frac{21}{144} \quad \boxed{M1} \text{ 6 cases and probs. multd. } \boxed{A1}
 \end{aligned}$$

$$\begin{aligned}
 P(X = 25) &= P(20+5, 5+20) \\
 &= 2 \cdot \frac{2}{144} \\
 &= \frac{4}{144} \quad \boxed{B1}
 \end{aligned}$$

$$(P(X < 25) = \frac{119}{144} \text{ is not required})$$

$$E(W) = \frac{21}{144} \cdot W + \frac{4}{144} \cdot 1 + 0 = \frac{21W + 4}{144} \quad \boxed{M1} \quad E(W) = \sum W P(W) \quad \boxed{A1}$$

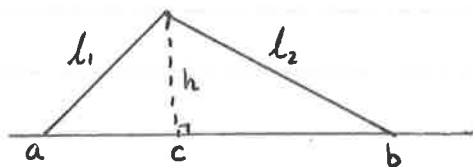
$$\text{Require } \frac{21W + 4}{144} - 1 < 0 \quad \boxed{M1}$$

$$\Rightarrow W < 7$$

$$\text{and max. integer value of } W \text{ is } \underline{6} \quad \boxed{A1}$$

7

SI 2014 Q13



$$\text{Area} = 1 = \frac{1}{2} (b-a) h \Rightarrow h = \frac{2}{b-a} \quad \boxed{\text{B1}}$$

$$\text{Grad. } l_1 \text{ is } \frac{2}{(b-a)(c-a)} \quad \boxed{\text{B1}}$$

$$\text{Eqn. } l_1 \text{ is } y = g(x) = \frac{2(x-a)}{(b-a)(c-a)} \quad \boxed{\text{M1}} \quad \boxed{\text{A1}} \quad \text{GIVEN ANSWER}$$

$$\text{Similarly, } l_2 \text{ has eqn } y = \frac{-2(x-b)}{(b-a)(b-c)} \quad \boxed{\text{M1}} \quad \boxed{\text{A1}}$$

6

$$(i) \quad E(x) = \int_a^c \frac{2}{(b-a)(c-a)} (x^2 - ax) dx + \int_c^b \frac{2}{(b-a)(b-c)} (bx - x^2) dx$$

$\boxed{\text{M1}}$ Attempted in two bits

$$= \frac{2}{(b-a)(c-a)} \left[\frac{x^3}{3} - ax^2 \right]_a^c + \frac{2}{(b-a)(b-c)} \left[\frac{bx^2}{2} - \frac{x^3}{3} \right]_c^b$$

$\boxed{\text{A1}} \quad \quad \quad \boxed{\text{A1}}$

Including correct signs and limits

$$= \frac{2}{b-a} \left\{ \frac{(c^3 - a^3) - \frac{a}{2}(c^2 - a^2)}{c-a} + \frac{\frac{b}{2}(b^2 - c^2) - (\frac{b^3 - c^3}{3})}{b-c} \right\}$$

$\boxed{\text{M1}}$ Factorisation method at some stage

$$= \frac{2}{b-a} \left\{ \frac{1}{3} (c^2 + ac + a^2) - \frac{1}{2} (ac + a^2) + \frac{1}{2} (b^2 + bc) - \frac{1}{3} (b^2 + bc + c^2) \right\}$$

$$= \frac{2}{b-a} \left\{ \frac{1}{6} (b^2 - a^2) + \frac{1}{6} (bc - ac) \right\} \quad \boxed{\text{M1}} \quad (b-a) \text{ factors identified}$$

$$= \frac{1}{3} (a+b+c) \quad \boxed{\text{A1}} \quad \text{GIVEN ANSWER}$$

6

(ii) If $c = \frac{1}{2}(a+b)$ then $m = c$ M1 Case identified
 (by symmetry) A1

2

If $c > \frac{1}{2}(a+b)$ then $a < m < c$ B1 Case identified

$$\left(\text{OR } \frac{c-a}{b-a} > \frac{1}{2} \right) \quad m \text{ g.b. } \frac{1}{2}(m-a) g(m) = \frac{1}{2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ M1 }$$

$$\Leftrightarrow \frac{\frac{1}{2}(m-a) \cdot 2(m-a)}{(b-a)(c-a)} = \frac{1}{2}$$

so that $\underline{\underline{m = a + \sqrt{\frac{1}{2}(b-a)(b-c)}}}$ A1

3

If $c < \frac{1}{2}(a+b)$ then $c < m < b$ B1 Case identified

$$\left(\text{OR } \frac{c-a}{b-a} < \frac{1}{2} \right) \text{ and } \underline{\underline{m = b - \sqrt{\frac{1}{2}(b-a)(b-c)}}} \quad \left. \begin{array}{l} \text{M1} \\ \text{A1} \end{array} \right\} \text{ Similarly}^* \quad 3$$

* May be implicit for 2nd region's working